Assignment 1. Solutions.

Problems. January 6.

1. Sec. 11.1, #10. $a_1 = -2$, $a_{n+1} = \frac{na_n}{n+1}$. Write out the first ten terms of the sequence.

   Solution.

   $a_1 = -2$, $a_2 = -\frac{2}{2}$, $a_3 = -\frac{2}{3}$, $a_4 = -\frac{2}{4}$, $a_5 = -\frac{2}{5}$, $a_6 = -\frac{2}{6}$, $a_7 = -\frac{2}{7}$, $a_8 = -\frac{2}{8}$, $a_9 = -\frac{2}{9}$, $a_{10} = -\frac{2}{10}$.

2. Sec. 11.1, #12. $a_1 = 2$, $a_2 = -1$, $a_{n+2} = \frac{a_{n+1}}{a_n}$. Write out the first ten terms of the sequence.

   Solution.

   $a_1 = 2$, $a_2 = -1$, $a_3 = -\frac{1}{2}$, $a_4 = \frac{1}{2}$, $a_5 = -1$, $a_6 = -2$, $a_7 = 2$, $a_8 = -1$, $a_9 = -\frac{1}{2}$, $a_{10} = \frac{1}{2}$.

3. Sec. 11.1, #16. Find a formula for the nth term of the sequence

   $$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \cdots$$

   (Reciprocals of squares of the positive integers with alternating signs).

   Solution.

   $$a_n = (-1)^{n-1}\frac{1}{n^2}, \quad n \geq 1.$$

4. Sec. 11.1, #20. Find a formula for the nth term of the sequence

   $$2, 6, 10, 14, 18, ...$$

   (Every other even positive integer).

   Solution.

   $$a_n = 4n - 2, \quad n \geq 1.$$

Problems. January 8.

1. Sec. 11.1, #28. Find the limit if it exists.

$$a_n = \frac{n+3}{n^2+5n+6}$$

Solution.

$$\lim_{n\to\infty} \frac{n+3}{n^2+5n+6} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{3}{n^2}}{1+\frac{5}{n}+\frac{6}{n^2}} = 0.$$

2. Sec. 11.1, #50. Find the limit if it exists.

$$a_n = \left(1-\frac{1}{n}\right)^n$$

Solution.

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1-\frac{1}{n}\right)}$$

The following limit can be computed using l'Hôpital's rule.

$$\lim_{n\to\infty} n\ln\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1-\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\left(1-\frac{1}{n}\right)} \Big/ \left(\frac{-1}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{-1}{1-\frac{1}{n}} = -1.$$

Therefore, the limit of a_n equals e^{-1}.

3. Sec. 11.1, #74. Find the limit if it exists.

$$a_n = n\left(1-\cos\frac{1}{n}\right)$$

Solution.

We will apply l'Hôpital's rule.

$$\lim_{n\to\infty} n\left(1-\cos\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\left(1-\cos\frac{1}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{-1}{n^2}\sin\frac{1}{n}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \sin\frac{1}{n} = 0.$$

4. Sec. 11.1, #82. Find the limit if it exists.

$$a_n = \frac{1}{\sqrt{n^2-1}} - \frac{1}{\sqrt{n^2+n}}$$

Solution.

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} - \frac{1}{\sqrt{n^2+n}} &= \lim_{n\to\infty} \frac{\sqrt{n^2+n}-\sqrt{n^2-1}}{\sqrt{n^2-1}\sqrt{n^2+n}} \\
&= \lim_{n\to\infty} \frac{\left(\sqrt{n^2+n}-\sqrt{n^2-1}\right)\left(\sqrt{n^2+n}+\sqrt{n^2-1}\right)}{\sqrt{n^2-1}\sqrt{n^2+n}\left(\sqrt{n^2+n}+\sqrt{n^2-1}\right)} \\
&= \lim_{n\to\infty} \frac{(n^2+n)-(n^2-1)}{\sqrt{n^2-1}\sqrt{n^2+n}\left(\sqrt{n^2+n}+\sqrt{n^2-1}\right)} \\
&= \lim_{n\to\infty} \frac{n+1}{\sqrt{n^2-1}\sqrt{n^2+n}\left(\sqrt{n^2+n}+\sqrt{n^2-1}\right)} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{1}{n^3}}{\sqrt{1-\frac{1}{n^2}}\sqrt{1+\frac{1}{n}}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n^2}}\right)} = 0.
\end{aligned}$$

Problems. January 11.

1. Determine if the sequence is nondecreasing and if it is bounded from above.

$$a_n = \frac{2n-1}{n+1}.$$

Solution.

$$a_n = \frac{2n-1}{n+1} = \frac{2n+2-3}{n+1} = \frac{2n+2}{n+1} - \frac{3}{n+1} = 2 - \frac{3}{n+1}.$$

The sequence is bounded from above since $a_n < 2$ for all n.

To show that it is nondecreasing, we observe that

$$\frac{1}{n+1} \geq \frac{1}{n+2},$$

and therefore

$$a_n = 2 - \frac{3}{n+1} \leq 2 - \frac{3}{n+2} = a_{n+1},$$

i.e.

$$a_n \leq a_{n+1}.$$

2. Write out the first few terms of the series to show how the series starts. If the series converges, find its sum.

$$\sum_{n=0}^{\infty}(-1)^n \frac{3^{n+1}}{5^n}.$$

Solution.

$$\sum_{n=0}^{\infty}(-1)^n \frac{3^{n+1}}{5^n} = 3 - \frac{3^2}{5} + \frac{3^3}{5^2} - \frac{3^4}{5^3} + \cdots.$$

This is a geometric series with $r = -3/5$. Since $|r| < 1$, the series converges and its sum equals

$$\frac{a}{1-r} = \frac{3}{1-(-3/5)} = \frac{15}{8}.$$

3. Express the number $1.\overline{23} = 1.23232323...$ as the ratio of two integers.

Solution.

Note that

$$1.\overline{23} = 1.23232323... = 1+\frac{23}{100}+\frac{23}{10000}+\frac{23}{1000000}+\cdots = 1+\sum_{n=1}^{\infty}\frac{23}{(100)^n}.$$

The series $\sum_{n=1}^{\infty}\frac{23}{(100)^n}$ is a geometric series whose first term is $23/100$ and the ratio is $1/100$. Therefore,

$$\sum_{n=1}^{\infty}\frac{23}{(100)^n} = \frac{23/100}{1-1/100} = \frac{23}{99},$$

and

$$1.\overline{23} = 1.23232323... = 1 + \frac{23}{99} = \frac{122}{99}.$$

4. Find the values of x for which the geometric series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{x^n}$$

converges. Also, find the sum of the series (as a function of x) for those values of x.

Solution.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{x^n}=\frac{2}{x}-\frac{2^2}{x^2}+\frac{2^3}{x^3}-\cdots$$

This is a geometric series with $a=2/x$ and $r=-2/x$. It is convergent when $|r|<1$, that is $2/|x|<1$, i.e. $|x|>2$. In the interval notation, $x\in(-\infty,-2)\cup(2,\infty)$. For these values of x the sum of the series equals

$$\frac{2/x}{1-(-2/x)}=\frac{2}{x+2}.$$

Problems. January 13.

1. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty}n\sin\left(\frac{1}{n}\right).$$

Solution.

We will use the n-th term test for divergence.

$$\lim_{n\to\infty}n\sin\left(\frac{1}{n}\right)=\lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}=\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Since $\lim_{n\to\infty}a_n\neq 0$, the series is divergent.

2. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\frac{1}{\sqrt{n}}}.$$

Solution.

We will use the n-th term test for divergence.

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)^{\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}e^{\frac{1}{\sqrt{n}}\ln\left(\frac{1}{n}\right)}=\lim_{n\to\infty}e^{-\frac{\ln n}{\sqrt{n}}}.$$

By l'Hôpital's rule,

$$\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}=\lim_{n\to\infty}\frac{1/n}{1/(2\sqrt{n})}=\lim_{n\to\infty}\frac{2}{\sqrt{n}}=0.$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)^{\frac{1}{\sqrt{n}}}=e^0=1.$$

Since this limit is not equal to zero, the series is divergent.

3. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty}\ln\frac{n+1}{n+2}.$$

Solution.

This is a telescoping series, since the n-th partial sum equals

$$s_n=\ln\frac{2}{3}+\ln\frac{3}{4}+\ln\frac{4}{5}+\cdots+\ln\frac{n}{n+1}+\ln\frac{n+1}{n+2}$$

$$=\ln 2-\ln 3+\ln 3-\ln 4+\ln 4-\ln 5+\cdots+\ln n-\ln(n+1)+\ln(n+1)-\ln(n+2)$$

$$=\ln 2-\ln(n+2).$$

We have

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}(\ln 2-\ln(n+2))=-\infty.$$

The latter is not a finite number, therefore the series diverges.

4. Use the integral test to determine whether the series

$$\sum_{n=1}^{\infty}\frac{n}{n^2+1}$$

converges or diverges.

Solution. Consider the function

$$f(x)=\frac{x}{x^2+1}.$$

This function is positive and continuous for $x \geq 1$. We need to check that it is decreasing.

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2} < 0,$$

if $x > 1$. Therefore, f is decreasing on the interval $(1, \infty)$.

Now we can apply the integral test.

$$\int_1^\infty \frac{x}{x^2+1} dx = \frac{1}{2} \ln(x^2+1) \Big|_1^\infty = \infty.$$

Since this integral is divergent, the original series is also divergent.

Problems. January 15.

1. Use the integral test to determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{e^{-n} \sin e^{-n}}{\cos e^{-n}}.$$

Solution.

Consider the function

$$f(x) = \frac{e^{-x} \sin e^{-x}}{\cos e^{-x}}.$$

We need to check whether this function is positive, continuous and decreasing.

Since $x \geq 1$, it follows that $e^{-x} \leq e^{-1} \approx 0.37 < \pi/2$. Therefore $\cos e^{-x}$ does not vanish for $x \geq 1$, and $f(x)$ is continuous for $x \geq 1$. Furthermore, $f(x) > 0$ for these values of x.

To see that f is decreasing, one can show that $f' < 0$. Alternatively, we can see that e^{-x} is a decreasing function, $\sin e^{-x}$ is also decreasing. $\cos e^{-x}$ is increasing, therefore $\frac{1}{\cos e^{-x}}$ is decreasing. It follows that f is a decreasing function, as it is a product of decreasing functions.

Since all the conditions of the integral test are satisfied, we will compute the integral

$$\int_1^\infty \frac{e^{-x} \sin e^{-x}}{\cos e^{-x}} dx = \left[u = e^{-x}, \quad du = -e^{-x} dx\right] = -\int_{1/e}^{0} \frac{\sin u}{\cos u} du$$

$$= \int_0^{1/e} \frac{\sin u}{\cos u} du = -\ln \cos u \Big|_0^{1/e} = -\ln\cos(1/e) + \ln\cos 0 = -\ln\cos(1/e).$$

The latter is a finite number, therefore the integral is convergent, and, by the integral test, the series is also convergent.

2. Use the integral test to determine whether the series is convergent or divergent

$$\sum_{n=3}^{\infty} \frac{1}{n \cdot \ln n \cdot \ln(\ln n)}.$$

Solution.

Consider the function

$$f(x) = \frac{1}{x \cdot \ln x \cdot \ln(\ln x)}.$$

If $x > 3$, then $\ln x > 1$, and therefore $\ln(\ln x) > 0$. Thus $f(x)$ is positive and continuous on the interval $(3, \infty)$. It is a decreasing function, since the denominator is a product of increasing functions. Alternatively, one can show that $f' < 0$.

Since all the assumptions of the integral test are satisfied, we look at the integral

$$\int_3^{\infty} \frac{dx}{x \cdot \ln x \cdot \ln(\ln x)} = \left[u = \ln x, du = \frac{dx}{x}\right] = \int_{\ln 3}^{\infty} \frac{du}{u \cdot \ln u}$$

$$= \left[v = \ln x, dv = \frac{dx}{x}\right] = \int_{\ln(\ln 3)}^{\infty} \frac{dv}{v} = \ln v \Big|_{\ln(\ln 3)}^{\infty} = \infty.$$

Since the latter integral is divergent, the series is also divergent, by the integral test.

3. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{\arctan n}{n^2 + \sqrt{n}}.$$

Solution.

This is a series with positive terms, and we will use the comparison test.

$$\frac{\arctan n}{n^2+\sqrt{n}} \le \frac{\pi/2}{n^2+\sqrt{n}} \le \frac{\pi/2}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{\pi/2}{n^2}$ is convergent as a p-series with $p = 2 > 1$, it follows that the original series is also convergent.

4. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{4+\cos n}{2n-1}.$$

Solution.

This is a series with positive terms, and we will use the comparison test. Since $\cos n \ge -1$, we have

$$\frac{4+\cos n}{2n-1} \ge \frac{4-1}{2n-1} \ge \frac{3}{2n}.$$

The series $\frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, and, therefore, divergent. Hence the original series is also divergent.

Problems. January 18.

1. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{2n^2+4n}{\sqrt{n^6+2n-3}}.$$

Solution.

We will us the limit comparison test. Let

$$a_n = \frac{2n^2+4n}{\sqrt{n^6+2n-3}}$$

and take

$$b_n = \frac{n^2}{\sqrt{n^6}} = \frac{1}{n}.$$

Then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{2n^2+4n}{\sqrt{n^6+2n-3}}\cdot n=\lim_{n\to\infty}\frac{2n^3+4n^2}{\sqrt{n^6+2n-3}}$$

$$\lim_{n\to\infty}\frac{2+\frac{4}{n}}{\sqrt{1+\frac{2}{n^5}-\frac{3}{n^6}}}=2.$$

The latter is a nonzero finite number. By the limit comparison test, either both $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ converge, or both diverge. But

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n}$$

is the divergent harmonic series. Therefore, the original series is also divergent.

2. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty}\frac{\ln n}{n^{1/3}}.$$

Solution.

We will use the comparison test

Since $\ln n\geq\ln 2$ for all $n\geq 2$, we have $\frac{\ln n}{n^{1/3}}\geq\frac{\ln 2}{n^{1/3}}$.

$\sum_{n=1}^{\infty}\frac{\ln 2}{n^{1/3}}$ is a divergent p-series, $p=1/3\leq 1$. Therefore, by the comparison test, the original series is divergent.

3. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty}\frac{\sqrt[n]{n}}{n^3}.$$

Solution.

We will use the limit comparison test. Let

$$a_n=\frac{\sqrt[n]{n}}{n^3}\quad\text{and}\quad b_n=\frac{1}{n^3}.$$

Consider the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{n^3} \cdot n^3 = \lim_{n\to\infty} \sqrt[n]{n} = 1.$$

The latter is a nonzero finite number. By the limit comparison test, either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, or both diverge. But

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

is convergent, as a p-series with $p = 3 > 1$. Therefore, the original series is also convergent.

4. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\pi^n (n+1)!}.$$

Solution.

We will use the ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\pi^{n+1}(n+2)!} \frac{\pi^n (n+1)!}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\pi^n \pi (n+1)!(n+2)} \frac{\pi^n (n+1)!}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \frac{1}{\pi(n+2)} = 0.$$

Since the limit is less than 1, the series is convergent by the ratio test.

Problems. January 20.

1. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{(3n)!}{3^n (n!)^3}.$$

Solution.

We will use the ratio test.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(3n+3)!}{3^{n+1}((n+1)!)^3}\frac{3^n(n!)^3}{(3n)!}$$

$$=\lim_{n\to\infty}\frac{(3n)!(3n+1)(3n+2)(3n+3)}{3^n3(n!(n+1))^3}\frac{3^n(n!)^3}{(3n)!}$$

$$=\lim_{n\to\infty}\frac{(3n+1)(3n+2)(3n+3)}{3(n+1)^3}$$

$$=\lim_{n\to\infty}\frac{(3+\frac{1}{n})(3+\frac{2}{n})(3+\frac{3}{n})}{3(1+\frac{1}{n})^3}=9>1.$$

By the ratio test the series is divergent.

2. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty}\frac{(\ln n)^{2n}}{n}.$$

Solution.

We will use the root test.

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{(\ln n)^2}{\sqrt[n]{n}}=\infty,$$

since $\ln n\to\infty$ and $\sqrt[n]{n}\to 1$, as $n\to\infty$.

By the root test, the series is divergent.

3. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty}\left(1-\frac{2}{n}\right)^{n^2}.$$

Solution.

We will use the root test.

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n=\lim_{n\to\infty}e^{n\ln\left(1-\frac{2}{n}\right)}$$

Applying l'Hôpital's rule we see that

$$\lim_{n\to\infty} n \ln\left(1-\frac{2}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1-\frac{2}{n}\right)}{1/n} = \lim_{n\to\infty} \frac{\frac{2}{n^2}}{\left(1-\frac{2}{n}\right)} \Big/ \left(\frac{-1}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{-2}{1-\frac{2}{n}} = -2.$$

Therefore,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = e^{-2} < 1.$$

By the root test, the series is convergent.

4. Test the series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{n^3+1}.$$

Solution.

In order to use the alternating series test, we need to check whether the sequence $u_n = \frac{n^2}{n^3+1}$ is increasing and has limit zero.

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0.$$

To show that u_n is decreasing, we will consider the function

$$f(x) = \frac{x^2}{x^3+1}.$$

Then

$$f'(x) = \frac{2x(x^3+1) - 3x^4}{(x^3+1)^2} = \frac{2x - x^4}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}.$$

The latter is negative, if $2 - x^3 < 0$, that is $x > \sqrt[3]{2}$. Therefore u_n is decreasing for $n \geq 2$.

By the alternating series test, the series is convergent.

Problems. January 22.

1. Test the series for absolute convergence, conditional convergence or divergence
$$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1}{n(\ln n)^2}.$$

   Solution.

   Let us test the series for absolute convergence. Consider the series
$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}.$$
   Here we can apply the integral test, since $f(x) = \frac{1}{x(\ln x)^2}$ is positive, continuous and decreasing for $x \geq 2$. Thus we need to compute
$$\int_2^{\infty}\frac{1}{x(\ln x)^2}dx = \left[u = \ln x, du = \frac{dx}{x}\right] = \int_{\ln 2}^{\infty}\frac{1}{u^2}du = -\frac{1}{u}\Big|_{\ln 2}^{\infty} = \frac{1}{\ln 2}.$$
   Since the latter integral is convergent, the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ is also convergent, according to the integral test. This means that the original series is absolutely convergent.

2. Test the series for absolute convergence, conditional convergence or divergence
$$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1}{n\ln n}.$$

   Solution.

   Let us first test the series for absolute convergence. Consider
$$\sum_{n=2}^{\infty}\frac{1}{n\ln n}.$$
   For this series we can use the integral test, since the function $f(x) = \frac{1}{x\ln x}$ is positive, continuous and decreasing for $x \geq 2$. We have
$$\int_2^{\infty}\frac{1}{x\ln x}dx = \left[u = \ln x, du = \frac{dx}{x}\right] = \int_{\ln 2}^{\infty}\frac{1}{u}du = \ln u\Big|_{\ln 2}^{\infty} = \infty.$$
   Since the latter integral is divergent, the series $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ is also divergent, according to the integral test. This means that the original series is not absolutely convergent.

On the other hand, the original series is convergent, according to the alternating series test. This follows from the fact that the sequence $u_n = \frac{1}{n \ln n}$ is decreasing and has limit zero.

Since the original series is convergent, but not absolutely convergent, it is conditionally convergent.

3. Test the series for absolute convergence, conditional convergence or divergence

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{1+e^{-n}}.$$

Solution.

Observe that

$$\lim_{n\to\infty}\frac{1}{1+e^{-n}} = 1.$$

Therefore, for odd n, the terms of $(-1)^{n-1}\frac{1}{1+e^{-n}}$ will tend to 1, and for even n, they will tend to -1. Thus the limit of $(-1)^{n-1}\frac{1}{1+e^{-n}}$ does not exist. By the n-th term test for divergence, the series is divergent.

4. Given

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+1}.$$

Check the hypotheses of the Alternating Series Test. How many terms of the series do we need to add in order to approximate the sum of the series to within 0.01?

Solution.

Consider

$$u_n = \frac{1}{n^2+1}.$$

This sequence is decreasing, since

$$u_n = \frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} = u_{n+1},$$

and

$$\lim_{n\to\infty}\frac{1}{n^2+1} = 0.$$

Therefore the hypotheses of the alternating series theorem are satisfied, and thus, the series is convergent.

In order to approximate the sum of the series to within 0.01, we need to find n such that $u_{n+1} < 0.01$. We have

$$\frac{1}{(n+1)^2+1} < 0.01,$$

$$\begin{aligned} 100 &< (n+1)^2+1, \\ 99 &< (n+1)^2, \\ \sqrt{99} &< n+1, \\ n &> \sqrt{99}-1. \end{aligned}$$

Therefore $n = 9$ is sufficient to achieve the given accuracy.

Problems. January 25.

For the given power series, find the radius of convergence and interval of convergence.

1.

$$\sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{(n+1)3^n}.$$

Solution.

We will use the ratio test.

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+2)3^{n+1}} \frac{(n+1)3^n}{|x|^n} = \lim_{n\to\infty} \frac{|x|(n+1)}{(n+2)3} = \frac{|x|}{3}$$

The series converges if the latter limit is less than 1. That is,

$$|x| < 3,$$

which means that $x \in (-3, 3)$. Thus, the radius of convergence equals $R = 3$.

Now test the endpoints of the interval $(-3, 3)$.

If $x = -3$, then the series becomes

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{(-3)^n}{(n+1)3^n} = -\sum_{n=0}^{\infty}\frac{1}{n+1}.$$

The latter is the harmonic series (one can see this after changing the index of summation $k = n+1$). Thus, divergent.

If $x = 3$, then the series becomes

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{3^n}{(n+1)3^n} = -\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n+1}.$$

The latter is the alternating harmonic series. It converges by the alternating series test.

Thus, the interval of convergence of the original power series is $(-3, 3]$, which means that $-3 < x \leq 3$.

2.

$$\sum_{n=1}^{\infty}\frac{(x+1)^n}{2^n\sqrt{n}}.$$

Solution.

We will use the ratio test.

$$\lim_{n\to\infty}\frac{|x+1|^{n+1}}{2^{n+1}\sqrt{n+1}}\frac{2^n\sqrt{n}}{|x+1|^n} = \lim_{n\to\infty}\frac{|x+1|\sqrt{n}}{2\sqrt{n+1}} = \frac{|x+1|}{2}\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = \frac{|x+1|}{2}$$

The series converges if the latter limit is less than 1. That is,

$$|x+1| < 2,$$

which means that

$$-2 < x+1 < 2,$$

$$-3 < x < 1,$$

$$x \in (-3, 1).$$

Thus, the radius of convergence equals $R = 2$.

Now test the endpoints of the interval $(-3, 1)$.

If $x = -3$, then the series becomes

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

The latter is an alternating series. Since $u_n = \frac{1}{\sqrt{n}}$ is decreasing and has limit zero, the series converges by the alternating series test.

If $x = 1$, then the series becomes

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

The latter is a p-series with $p = 1/2 \leq 1$. Thus, divergent.

So, the interval of convergence of the original power series is $[-3, 1)$, which means that $-3 \leq x < 1$.

3.
$$\sum_{n=1}^{\infty} \frac{3^{2n-1}(x-2)^n}{n!}.$$

Solution.

We will use the ratio test.

$$\lim_{n\to\infty} \frac{3^{2n+1}|x-2|^{n+1}}{(n+1)!} \frac{n!}{3^{2n-1}|x-2|^n} = \lim_{n\to\infty} \frac{9|x-2|}{n+1} = 0$$

for all real x. Since the limit is less than 1 for all real x, it means that the series is convergent for all real x. Therefore, the radius of convergence is $R = \infty$, and the interval of convergence is $\mathbb{R} = (-\infty, \infty)$.

4.
$$\sum_{n=1}^{\infty} \frac{(2n)!(2x+3)^n}{n5^n}.$$

Solution.

We will use the ratio test.

$$\lim_{n\to\infty} \frac{(2n+2)!|2x+3|^{n+1}}{(n+1)5^{n+1}} \frac{n5^n}{(2n)!(2x+3)^n}$$

$$= \lim_{n\to\infty} \frac{(2n)!(2n+1)(2n+2)|2x+3|^{n+1}}{(n+1)5^{n+1}} \frac{n5^n}{(2n)!|2x+3|^n}$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)n|2x+3|}{5(n+1)}$$

$$= \lim_{n\to\infty} |2x+3| \frac{(2n+1)(2n+2)}{5} \frac{n}{n+1}$$

If $2x+3=0$, that is $x=-3/2$, then the latter expression is zero. Therefore, the limit is also zero. Since it is less than 1, the series is convergent at $x=-3/2$.

If $2x+3 \neq 0$, then the limit equals ∞. This means that for all $x \neq -3/2$ the series is divergent. Thus the radius of convergence is $R=0$, and the series converges only at one point $x=-3/2$.